Another Interesting Result for the Number *e*

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There's a math teaser that asks: which is bigger e^{π} or π^{e}?

The answer is not immediately obvious but one might arrive at the correct answer by the following reasoning. If I have a number x^y with x and y real, positive numbers then, generally speaking, x^y increases faster as y increases than if x increases. Therefore, it would seem that because $\pi > e$ that e^{π} is greater than π^{e}. This, in fact, is the correct result as a simple calculation will verify.

Looking at this teaser one might go one step further and ask if there is any $\Delta > 0$ such that $e^{e+\Delta} \leq (e+\Delta)^e$?

Well, the answer is no! This provides further evidence that the reasoning above is sound. But is the reasoning correct?

Let's take the question one step further. Consider the relationship

$$(1) \qquad x^{(x+\Delta)} \leq (x+\Delta)^x$$

where $x > 1$ and $\Delta > 0$ are real numbers. For a given x are there any values of $\Delta > 0$ such that relationship (1) holds? As discussed above, the answer is no at least if $x=e$.

Here's the complete answer:

- For $1 < x < e$, there exists Δ_x such that relationship (1) holds if and only if $0 < \Delta < \Delta_x$.
- For $x=e$ only $\Delta=0$ satisfies relationship (1).
- If we allow $\Delta < 0$ then for $x > e$, there exists $\Delta_x < x$ such that relationship (1) holds if and only if $-\Delta_x < \Delta < 0$.

An interesting aside: if $x=2$ then $\Delta_x = x$. In fact, for $2 < x < e$, $\Delta_x < x$ while for $0 < x < 2$, $\Delta_x > x$.

<u>Proof for the case $x=e$</u>

Let us show that $e^{e+\Delta} > (e+\Delta)^e$ for all $\Delta > 0$ by taking the log of both sides (noting that the log is positive) and expanding the right hand side in a Taylor series around e.

$$\begin{aligned} (e+\Delta) \ln e \quad &> e \ln(e+\Delta) \\ &= e\,[1 + (\Delta/e) - \tfrac{1}{2}\,(\Delta/e)^2 + \tfrac{1}{3}\,(\Delta/e)^3 - \ldots] \\ &= e+\Delta - \Delta^2/e\,[\tfrac{1}{2} - \tfrac{1}{3}(\Delta/e) + \ldots] \end{aligned}$$

Since $\ln e = 1$, we can subtract $e+\Delta$ from both sides and see that this relationship implies that $0 > -\Delta^2/e\,[\tfrac{1}{2} - \tfrac{1}{3}(\Delta/e) + \ldots]$ which is clearly true since, as $\Delta \rightarrow 0$ the term in the bracket approaches ½.